A note on lower bounds for hypergraph Ramsey numbers

David Conlon*

Abstract

We improve upon the lower bound for 3-colour hypergraph Ramsey numbers, showing, in the 3-uniform case, that

$$r_3(l,l,l) \geq 2^{l^{c \log\log l}}.$$

The old bound, due to Erdős and Hajnal, was

$$r_3(l,l,l) \geq 2^{cl^2 \log^2 l}.$$

1 Introduction

The hypergraph Ramsey number $r_k(l,l)$ is the smallest number n such that, in any 2-colouring of the complete k-uniform hypergraph K_n^k, there exists a monochromatic K_l^k. That these numbers exist is exactly the statement of Ramsey's famous theorem [7].

These numbers were studied in detail by Erdős and Rado [5], who showed that

$$r_k(l,l) \leq 2^{2^{\cdot^{\cdot^{\cdot^{2^{cl}}}}}},$$

where the tower is of height k and c is a constant that depends on k.

For the lower bound, there is an ingenious construction, due to Erdős and Hajnal ([6], [4],[3]), which allows one to show, for $k \geq 3$, that

$$r_k(l,l) \geq 2^{2^{\cdot^{\cdot^{\cdot^{2^{cl^2}}}}}},$$

where this time the tower is of height $k-1$ and c is another constant depending on k. Their construction uses a so-called stepping-up lemma, which allows one to construct counterexamples of higher uniformity from ones of lower uniformity, effectively giving an extra exponential each time we apply it to move up to a higher uniformity. Unfortunately, it does not allow one to step up graph counterexamples to 3-uniform counterexamples, and it is here that we lose out on the single exponential by which the towers differ. Instead, we have to start from a different 3-uniform counterexample, the simple probabilistic one, which yields

$$r_3(l,l) \geq 2^{cl^2},$$

and use that to step up.

*St John's College, Cambridge, CB2 1TP, United Kingdom. E-mail: D.Conlon@dpmms.cam.ac.uk

Erdős was obviously very fond of this problem, offering \$500 for the person who could close the gap between the upper and the lower bound. As yet, there has been no progress, in the 2-colour case, beyond the bounds we have given above. However, the number of colours seems to matter quite a lot in this problem. Erdős and Hajnal were already aware (again, see [6]) that a variant on their methods could produce a counterexample showing that indeed

$$r_k(l,l,l,l) \geq 2^{2^{\cdot^{\cdot^{\cdot^{2^{cl}}}}}},$$

where now the tower has the correct height k and again c depends on k.

Naturally, in the 3-colour case, one would also expect some little improvement, and Erdős and Hajnal provided just such a result (unpublished, see [1], though the reader may consult [2] for an earlier attempt), showing that

$$r_3(l,l,l) \geq 2^{cl^2 \log^2 l}.$$

It is this case that we will look at in this paper, showing that the bound may be improved rather more substantially to

Theorem 1

$$r_3(l,l,l) \geq 2^{l^{c \log \log l}}.$$

Our method is in the stepping-up lemma tradition. It differs, however, from the lemmas proved in the past in that we make explicit use of the probabilistic method in our construction. A rough idea of the proof is that we choose a very dense graph G containing no cliques of size l. We then step up to a dense 3-uniform graph H and 2-colour it. The specific form of the 2-colouring implies that we cannot contain a monochromatic 3-uniform $(l+1)$-clique without G containing an l-clique. The complement of H is coloured with the third colour. It is the step up of the complement of G, and we show, by an involved argument, that this sparse graph can be chosen in such a way that the third colour (the step-up of this graph) does not contain an $(l+1)$-clique. It is this part of the argument which is new and facilitates our improvement.

Once we have the 3-uniform case, we can then apply the stepping-up lemma of Erdős and Hajnal, which we state as

Theorem 2 *If $k \geq 3$ and $r_k(l) \geq n$, then $r_{k+1}(2l+k-4) \geq 2^n$.*

to give the following theorem

Theorem 3

$$r_k(l,l,l) \geq 2^{2^{\cdot^{\cdot^{\cdot^{2^{l^{c \log \log l}}}}}}},$$

where the tower is of height k and the constant c depends on k.

2 Proof of Theorem 1

Note that, throughout this section, whenever we use the term log we mean log taken to the base 2.

Let G be a graph on n vertices which does not contain a clique of size l . We are going to consider the complete 3-uniform hypergraph on the set

$$T = \{(\gamma_1, \cdots, \gamma_n) : \gamma_i = 0 \text{ or } 1\}.$$

If $\epsilon = (\gamma_1, \cdots, \gamma_n)$, $\epsilon' = (\gamma'_1, \cdots, \gamma'_n)$ and $\epsilon \neq \epsilon'$, define

$$\delta(\epsilon, \epsilon') = \max\{i : \gamma_i \neq \gamma'_i\},$$

that is, $\delta(\epsilon, \epsilon')$ is the largest component at which they differ. Given this, we can define an ordering on T, saying that

$$\epsilon < \epsilon' \text{ if } \gamma_i = 0, \gamma'_i = 1,$$
$$\epsilon' < \epsilon \text{ if } \gamma_i = 1, \gamma'_i = 0.$$

Equivalently, associate to any ϵ the number $b(\epsilon) = \sum_{i=1}^{n} \gamma_i 2^{i-1}$. The ordering then says simply that $\epsilon < \epsilon'$ iff $b(\epsilon) < b(\epsilon')$.

We will do well to note the following two properties of the function δ:

(a) if $\epsilon_1 < \epsilon_2 < \epsilon_3$, then $\delta(\epsilon_1, \epsilon_2) \neq \delta(\epsilon_2, \epsilon_3)$;
(b) if $\epsilon_1 < \epsilon_2 < \cdots < \epsilon_m$, then $\delta(\epsilon_1, \epsilon_m) = \max_{1 \leq i \leq m-1} \delta(\epsilon_i, \epsilon_{i+1})$.

Now, consider the complete 3-uniform hypergraph H on the set T. If $\epsilon_1 < \epsilon_2 < \epsilon_3$, let $\delta_1 = \delta(\epsilon_1, \epsilon_2)$ and $\delta_2 = \delta(\epsilon_2, \epsilon_3)$. Note that, by property (a) above, δ_1 and δ_2 are not equal. Colour the edge $\{\epsilon_1, \epsilon_2, \epsilon_3\}$ as follows:

C_1, if $\{\delta_1, \delta_2\} \in e(G)$ and $\delta_1 < \delta_2$;
C_2, if $\{\delta_1, \delta_2\} \in e(G)$ and $\delta_1 > \delta_2$;
C_3, if $\{\delta_1, \delta_2\} \notin e(G)$.

Suppose that C_1 contains a clique $\{\epsilon_1, \cdots, \epsilon_{l+1}\}_<$ of size $l+1$. For $1 \leq i \leq l$, let $\delta_i = \delta(\epsilon_i, \epsilon_{i+1})$. Note that the δ_i form a monotonically increasing sequence, that is $\delta_1 < \delta_2 < \cdots < \delta_l$. Also, note that since, for any $1 \leq i < j \leq l$, $\{\epsilon_i, \epsilon_{i+1}, \epsilon_{j+1}\} \in C_1$, we have, by property (b) above, that $\delta(\epsilon_{i+1}, \epsilon_{j+1}) = \delta_j$, and thus $\{\delta_i, \delta_j\} \in e(G)$. Therefore, the set $\{\delta_1, \cdots, \delta_l\}$ must form a clique of size l in G. But we have chosen G so as not to contain such a clique, so we have a contradiction. Similarly, C_2 cannot contain a clique of size $l+1$.

For C_3, assume again that we have a monochromatic clique $\{\epsilon_1, \cdots, \epsilon_{l+1}\}_<$ of size $l+1$, and, for $1 \leq i \leq l$, let $\delta_i = \delta(\epsilon_i, \epsilon_{i+1})$. Not only can we no longer guarantee that these form a monotonic sequence, but we can no longer guarantee that they are distinct. Suppose, however, that there are d distinct values of Δ. We will consider the graph J on the vertex set $\{\Delta_1, \cdots, \Delta_d\}$ with edge set given by all those $\{\Delta_i, \Delta_j\}$ such that there exists $\epsilon_r < \epsilon_s < \epsilon_t$ with $\{\Delta_i, \Delta_j\} = \{\delta(\epsilon_r, \epsilon_s), \delta(\epsilon_s, \epsilon_t)\}$. Understanding the properties of these graphs is essential because these graphs are exactly the ones that we will need to avoid in the complement of G in order to avoid stepping-up to a complete graph.

How many edges are there in J? To begin, note that Δ_1 is joined to all other Δ, so we have at least $d-1$ edges. Suppose that the one occurrence of Δ_1 is at δ_{i_1}. For the sake of later brevity,

note that we may sometimes refer to these as $\Delta_{1,1}$ and $\delta_{i_{1,1}}$ respectively. Now, let $\Delta_{2,1}$ be the largest δ_j, at $\delta_{i_{2,1}}$ say, to the left of $\delta_{i_{1,1}}$ (that is with $j < i_{1,1}$), a region which we will denote by $R_{2,1}$. Similarly, let $\Delta_{2,2}$ be the largest δ, occurring at $\delta_{i_{2,2}}$, in the region $R_{2,2}$ which is to the right of $\delta_{i_{1,1}}$. $\Delta_{2,1}$ (resp. $\Delta_{2,2}$) must then be joined to every δ which is to the left (resp. right) of $\delta_{i_{1,1}}$. Therefore, since there must be representatives of all remaining Δ amongst these δ, we see that between $\Delta_{2,1}$ and $\Delta_{2,2}$, they must have $d - t_2$ neighbours, where t_2 is the number of distinct Δ amongst $\Delta_{i_{1,1}}, \Delta_{i_{2,1}}, \Delta_{i_{2,2}}$.

Continuing inductively, suppose that we have the collection of $\Delta_{a,b}$ for all $1 \leq a \leq i-1$ and all $1 \leq b \leq 2^{a-1}$. This collection, consisting of at most $2^{i-1} - 1$ of the δ, partitions the δ into at most 2^{i-1} regions, which, starting from the left with δ_1 and working towards δ_l on the right, we denote by $R_{i,1}, R_{i,2}, \cdots, R_{i,2^{i-1}}$. Choose, within each region $R_{i,j}$, the largest δ, which we denote by $\Delta_{i,j}$. Each of these is necessarily distinct from all $\Delta_{a,b}$ with $1 \leq a \leq i-1$. Let t_i be the number of distinct δ given by the list of numbers $\Delta_{a,b}$ for $1 \leq a \leq i$ and $1 \leq b \leq 2^{a-1}$. Then, since each of the remaining Δ must lie in one of the regions $R_{i,j}$, we see that at least one of $\Delta_{i,1}, \cdots, \Delta_{i,2^{i-1}}$ must be connected to each of the $d - t_i$ remaining Δ.

We continue this process until we run out of representatives, that is until the mth step, when $t_m = d$. Note that there must be such an m, since we must add at least one new Δ class at each step. Note also that $m \geq \log(l+1)$. This is because, unless we have used up all of the δ in our process there will always be some extra distinct representatives remaining to consider. So we must have that $2^m - 1$, which is the maximum number of δs considered at step m, is at least as large as l. Consequently, as $d \geq m$, we also have that $d \geq \log(l+1)$.

Now, overall, we have

$$(d - t_1) + \cdots + (d - t_m) = dm - (t_1 + \cdots + t_m)$$

edges. To get a lower bound on this, we need to have upper bounds for each of the t_i. A straightforward upper bound for t_i, following from the fact that $t_i \geq t_{i-1} + 1$, is $t_i \leq d - m + i$. For small i we can do better, since there we know that $t_i \leq 2^i - 1$. Therefore, letting $i_0 = \log(d - m + 1)$, we have

$$\begin{aligned} t_1 + \cdots + t_m &= \sum_{i=1}^{i_0} t_i + \sum_{i=i_0+1}^{m} t_i \\ &\leq 2(d - m + 1) + \sum_{i=i_0+1}^{m} (d - m + i) \\ &= 2(d - m + 1) + \sum_{j=0}^{m-i_0-1} (d - j) \\ &= 2(d - m + 1) + d(m - i_0) - \frac{(m - i_0)(m - i_0 - 1)}{2}. \end{aligned}$$

Subtracting this from dm, we see that the total number of edges is at least

$$di_0 + \frac{(m - i_0)(m - i_0 - 1)}{2} - 2(d - m + 1).$$

Now, if $d - m + 1 \geq \frac{1}{2}\log(l+1)$, we have, since $i_0 = \log(d - m + 1)$, that this is greater than

$$d(\log\log(l+1) - 3).$$

If, on the other hand, $d - m + 1 \le \frac{1}{2}\log(l+1)$, we have that $m \ge d + 1 - \frac{1}{2}\log(l+1) \ge d/2 + 1$ (recall that $d \ge \log(l+1)$) and, therefore, the number of edges is at least

$$\frac{1}{8}(d + 2 - 2\log\log(l+1))(d - \log\log(l+1)) - 2d \ge \frac{1}{10} d\log(l+1),$$

for l large. So, in any case, for l sufficiently large, we have that the number of edges is at least

$$\frac{1}{10} d\log\log(l+1).$$

Now, for any graph J, let J' be the graph formed by the process of joining $\Delta_{i,j}$ to all Δ that have representatives in the region $R_{i,j}$. If at any stage we find that we have Δ_{i,j_1} and Δ_{i,j_2}, both of which are joined to the same Δ, then we remove one of the edges arbitrarily, eventually forming a graph J''. Every graph J must contain such a graph. In fact, above, it is the minimum number of edges in an associated J'' that we have counted. The question we must now ask is, how many distinct J'', up to isomorphism, are there, given that we have a certain d?

Consider the set of vertices $V = \{v_1, \cdots, v_d\}$. Choose the vertex v_1 and join it to all other vertices. Now consider the set $V \backslash \{v_1\}$. Up to isomorphism there are at most d different ways to partition this set into two sets $V_{2,1}$ and $V_{2,2}$, say. Now choose a vertex in each set, say $v_{2,1}$ and $v_{2,2}$, and join each to all other vertices in their respective sets. Consider, in turn, the sets $V_{2,1}\backslash\{v_{2,1}\}$ and $V_{2,2}\backslash\{v_{2,2}\}$, and partition each into two sets $V_{3,1}, V_{3,2}$ and $V_{3,3}, V_{3,4}$ respectively. Again, up to isomorphism there are at most d ways to partition each of the sets. So, overall, we have at most d^3 non-isomorphic classes at this stage.

Continue in the same way. At the $i-1$st stage, we have sets $V_{i-1,1}, \cdots, V_{i-1,2^{i-2}}$. Choose, in each set $V_{i-1,j}$, a vertex $v_{i-1,j}$ and join it to every other vertex in the set. Then partition each set $V_{i-1,j}\backslash\{v_{i-1,j}\}$ into two sets. As always, this can be done, up to isomorphism in at most d ways. This process stops when we run out of vertices.

Note that at each step we choose a vertex and then partition an associated set. Since there are at most d vertices and the number of ways to partition any set is at most d, we conclude that the number of non-isomorphic graphs J'' is at most d^d. (This is, of course, quite a rough estimate, but it is relatively easy to prove and perfectly sufficient for our purposes.)

We are finally ready to pick the graph G. Recall that, for the first two colours not to contain a 3-clique of size $l+1$, we need to choose G so as not to contain a clique of size l. Moreover, for the last colour not to contain a 3-clique of size $l+1$, it is sufficient that the complement of G, denoted by $\overline{G}$, does not contain any of the graphs J''.

We are going to fix $n = l^{c\log\log l}$, where c is a constant to be determined, and choose edges with probability $p = 1 - \frac{\log l \log\log l}{l}$. The expected number of cliques of size l in G is then

$$\begin{aligned} p^{\binom{l}{2}}\binom{n}{l} &= \left(1 - \frac{\log l\log\log l}{l}\right)^{\binom{l}{2}} l^{cl\log\log l} \\ &\le e^{-\frac{1}{2}l\log l\log\log l} e^{cl\log l\log\log l} \\ &\le e^{-\frac{1}{4}l\log l\log\log l}, \end{aligned}$$

if we take $c \le 1/4$.

On the other hand, the expected number of graphs J'' of order d that we can expect to find in $\overline{G}$ is at most

$$\begin{aligned} d^d(1-p)^{\frac{1}{10}d\log\log(l+1)}n^d &\leq \left(\frac{\log l \log\log l}{l}\right)^{\frac{1}{10}d\log\log(l+1)}(dn)^d \\ &\leq e^{-\frac{1}{20}d\log l\log\log(l+1)}l^{2cd\log\log l} \\ &\leq e^{-\frac{1}{40}d\log l\log\log l}, \end{aligned}$$

if we take $c \leq 1/80$ and l sufficiently large.

Adding over the expected number of cliques in G and the expected number of copies of graphs J'' in $\overline{G}$ for all l possible values of d, we find that, for l sufficiently large, the expected value of all such graphs is less than one. We can therefore choose our graph G in such a way that it does not itself contain a clique of size l and its complement $\overline{G}$ does not contain any of the graphs J''. The result follows.

References

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